# A discrete duality finite volume method for elliptic problems with corner singularities

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**Abstract**

We focus on the Discrete Duality Finite Volume (DDFV) method whose particularity is to allow the use of unstructured or nonconforming meshes. We deal with the Laplacian problem on nonconvex domains. We show how appropriate refinement conditions on the diamond mesh lead to an optimal order of convergence as for smooth solutions. These theoretical results are illustrated by some numerical applications.

**Key words** : finite volumes, discrete duality, corner singularities.

## 1 Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^2$ whose polygonal boundary is denoted by $\Gamma$. We note by $H^m(\Omega)$, with $m \in \{1,2\}$, the usual Sobolev spaces with norm $\|.\|_{H^m(\Omega)}$ and $L^p(\Omega)$, with $p \geq 1$, the usual Lebesgue spaces with norm $\|.\|_{L^p(\Omega)}$. We consider the following problem: given $f \in L^2(\Omega)$, let $\phi \in H^1(\Omega)$ be the variational solution of the Laplace equation with Dirichlet homogeneous boundary conditions:

$$\left\{ \begin{array}{rcl} -\Delta\phi & = & f \;\text{ in } \Omega, \\ \phi & = & 0 \;\text{ on } \Gamma, \end{array} \right. \tag{1}$$

or Neumann homogeneous boundary conditions:

$$\left\{ \begin{array}{rcl} -\Delta\phi & = & f \;\text{ in } \Omega, \\ \boldsymbol{\nabla}\phi \cdot \mathbf{n} & = & 0 \;\text{ on } \Gamma, \\ \int_\Omega \phi \, d\mathbf{x} & = & 0. \end{array} \right. \tag{2}$$

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A necessary condition for existence of a solution to (2) is given by $\int_\Omega f(\mathbf{x})d\mathbf{x} = 0$. It is well known that when $\Omega$ is convex, the solutions of the problems (1) and (2) are smooth [CIA 78] and belong to $H^2(\Omega)$. However, $\Omega$ is rarely convex in practice. Therefore, many studies have been conducted on polygonal domains presenting reentrant corners and we know [GRI 92, WAH 84, LAA 57, DAU 88, KS 87] that the solutions of (1) and (2) have singularities which lead to a loss of regularity near the nonsmooth parts of the boundary (here, the reentrant corners), even if the datum $f$ is smooth on $\overline{\Omega}$.
More precisely, when $\Omega$ has at least one angle $\omega_c \in ]\pi, 2\pi[$, associated to a corner noted by $c$, then the solution $\phi$ of the problems (1) and (2) can be rewritten as:

$$\phi = \widetilde{\phi} + \sum_{\omega_c > \pi} \nu_c \, \phi_c, \tag{3}$$

where $\widetilde{\phi} \in H^2(\Omega)$ is the smooth part, $\phi_c \notin H^2(\Omega)$ is the singular part associated to the corner $c$ whose angle $\omega_c$ belongs to $\in ]\pi, 2\pi[$ and $\nu_c$ is a real-valued number. In this case, according to [GRI 92, KON 67, BR 72, KS 87], the singular part $\phi_c$ is defined in polar coordinates $(r_c, \theta_c)$ as:

$$\phi_c(r_c, \theta_c) = \eta(r_c)\; r^{\frac{\pi}{\omega_c}} \sin\left(\frac{\pi\theta_c}{\omega_c}\right) \text{ for the problem (1)}, \tag{4}$$

and

$$\phi_c(r_c, \theta_c) = \eta(r_c)\; r^{\frac{\pi}{\omega_c}} \cos\left(\frac{\pi\theta_c}{\omega_c}\right) \text{ for the problem (2)}, \tag{5}$$

where $\eta(r_c) = 1$ in a neighborhood of the corner $c$ and 0 otherwise. Consequently, we notice that, for any neighborhood $V_c$ of the corner $c$, the solution $\phi$ belongs to $H^2(\Omega \setminus V_c)$ for both problems. More precisely, we know [GRI 92] that $\phi$ belongs to $H^{1+\frac{\pi}{\omega}-\epsilon}(\Omega)$ with $\epsilon > 0$ and $\omega = \max\limits_{\omega_c > \pi} \omega_c$.
In what follows, we assume that, without loss of generality, $\Omega$ has an unique corner such that $\omega > \pi$ whose vertex S is located at the origin (0,0). Such a configuration is displayed in figure 1.

Now, we introduce a family called weighted Sobolev spaces $H^{m,\alpha}(\Omega)$ with $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \geq 0$ such that

$$H^{m,\alpha}(\Omega) = \left\{ \phi \in H^{m-1}(\Omega) : |\phi|^2_{H^{m,\alpha}(\Omega)} = \sum_{|\beta|=m} \|r^\alpha D^\beta \phi\|^2_{L^2(\Omega)} < +\infty \right\},$$

where $r := r(\mathbf{x}) = d(\mathbf{x}, S)$ is the distance from $\mathbf{x} \in \Omega$ to the origin $S$. In addition, for $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1[$, we define a norm on $H^{m,\alpha}(\Omega)$ by

$$\|\phi\|^2_{H^{m,\alpha}(\Omega)} = \|\phi\|^2_{H^{m-1}(\Omega)} + \ |\phi|^2_{H^{m,\alpha}(\Omega)}.$$

We easily check that $H^2(\Omega) \subset H^{2,\alpha}(\Omega)$ and that the functions $\phi_c$ defined in (4) and (5) belong to $H^{2,\alpha}(\Omega)$ under the condition $\alpha \in \left]1 - \frac{\pi}{\omega}, \frac{1}{2}\right[$. Therefore, the solution $\phi \in H^1(\Omega)$ of each problem (1) and (2) belongs to $H^{2,\alpha}(\Omega)$ when $\Omega$ is nonconvex. This loss of regularity leads to a loss of accuracy near the corner for the standard

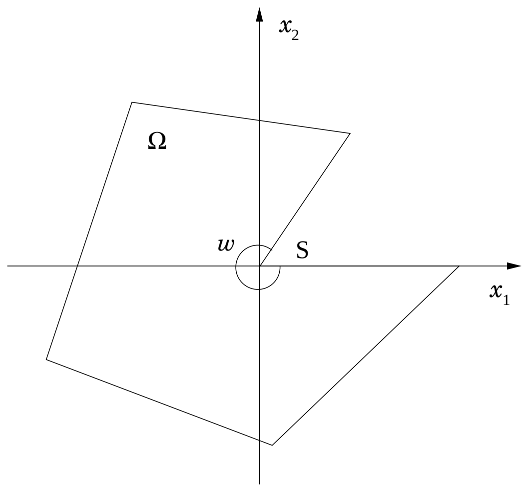


Figure 1: The domain $\Omega$ with only one corner.

discretization techniques. Indeed, [GRI 92, WAH 84, LAA 57, BKP 79] have shown that a quasi-uniform sequence of triangulations of $\Omega$ will not lead to optimal rates of convergence for the Galerkin approximation $\phi_h$ of the solution. Moreover, [DNT 02] numerically observes a loss of accuracy for the cell-centered finite volume method [EGH 00, HEI 94, RP 80], and for the conforming and nonconforming [BR 87, CAI 91, CHA 02, CHA 99] finite volume-element methods (called also box methods) which combine the finite element methods and the finite volume methods. At last, [CL 05] have proved theoretically the loss of accuracy of a finite volume-element method.

In fact, there is an extensive literature about the corner singularities (there exists also a lot of works on edge singularities [AN 98] in 3D or other weighted Sobolev spaces [BNZ 05] for example, but here we focus on corner singularities in 2D). The studies on the loss of regularity for the Laplacian problem are mainly based on a theoretical or numerical, finite element based, point of view, but, to our knowledge, this problem has not been much studied for the finite volume methods (see [HEI 94, DNT 02]), although these methods are interesting for the approximation of various physical phenomena (computational fluid dynamics or convection-diffusion problems for example).

In what follows, we are interested in a finite volume method, called the discrete duality finite volume (DDFV) method [DO 05, DDO 05]. The interest of this method is its ability to deal with arbitrary polygonal meshes such as nonconforming meshes or unstructured meshes without constraints of orthogonality, such as those used by [EGH 00] for example. The price to pay for that is the addition of unknowns. Therefore, for the Laplacian problem, the unknowns are located at the barycenters and at the vertices of the mesh, since the Laplacian equation is integrated both on the cells of the mesh (called primal mesh) and on a second mesh, called dual mesh, whose cells are centered at the vertices of the primal mesh.

We have also remarked a loss of accuracy for the Laplacian problem discretized by the DDFV method in the presence of reentrant corners (see [DDO 07]). That is why we study here the ability of an appropriate local refinement to restore optimal convergence for this finite volume method, like it was shown for the finite elements methods in [RAU 78, GRI 85]. In addition, the same techniques of refinement have succeeded (see [DNT 02]) on the cell-centered method, and on conforming and nonconforming finite volume-element methods. Therefore, in order to obtain the corresponding error estimates, we combine the error analysis of the DDFV scheme given in [DO 05] for smooth solutions of the Laplacian problem, with the error analysis of the Galerkin method for nonsmooth solutions described in [GRI 85, RAU 78] in addition to some ideas of [DNT 02]. The main difference for the DDFV method is that the analysis is given on a third mesh, called diamond mesh, which will be specified later, instead of on the primal mesh as usual [GRI 85, RAU 78, DNT 02]. This paper is organized as follows: in Section 2, we explain the construction of the primal, dual and diamond meshes and we present the associated notations. Then, we detail the definitions of the discrete differential operators used to discretize the Laplace operator and the discrete scalar products, and we recall the finite volume scheme for the Laplacian problem obtained in [DO 05]. Section 3 is devoted to the error analysis for nonsmooth solutions. We show how appropriate refinement conditions on the diamond mesh lead to an optimal order of convergence as for smooth solutions. These theoretical results are illustrated in Section 4 by some numerical results: firstly, on unstructured meshes without refinement and secondly on structured and unstructured meshes with appropriate refinement near the reentrant corner.

# 2 Definitions, notations and discrete schemes for the Laplacian

## 2.1 Construction of the primal mesh

We consider a first partition of $\Omega$ (named primal mesh) composed of elements $T_i$, with $i \in [1, I]$, supposed to be convex polygons. With each element $T_i$ of the mesh, we associate the center of gravity $G_i$. The area of $T_i$ is denoted by $|T_i|$. We denote by $J$ the total number of edges of this mesh and $J^\Gamma$ the number of these edges which are located on the boundary $\Gamma$ and we associate with each of these boundary edges its midpoint, also denoted by $G_i$ with $i \in [I+1, I+J^\Gamma]$.

## 2.2 Construction of the dual mesh

Further, we denote by $S_k$, with $k \in [1, K]$, the nodes of the polygons of the primal mesh. To each of these points, we associate a polygon denoted by $P_k$, obtained by joining the points $G_i$ associated to the elements of the primal mesh (and possibly to the midpoints of the boundary sides) of which $S_k$ is a vertex, to the midpoints of the edges of which $S_k$ is an extremity. The cells $P_k$ constitute a second partition of $\Omega$, referenced as dual mesh. The area of $P_k$ is denoted by $|P_k|$. Figure 2 displays an example of primal mesh and its associated dual mesh.
Moreover, we suppose that the set $[1, K]$ is ordered so that $k \in [1, K - J^\Gamma]$ if $S_k$ is not on $\Gamma$ and $k \in [K - J^\Gamma + 1, K]$ for nodes $S_k$ belonging to $\Gamma$.

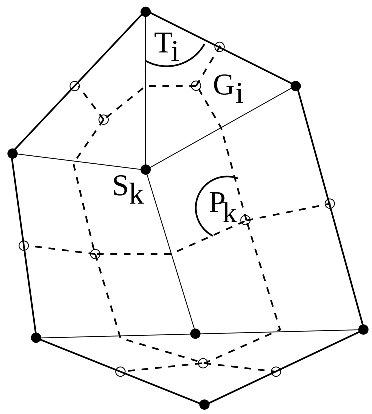


Figure 2: An example of a primal mesh and its associated dual mesh.

## 2.3 Construction of the diamond mesh

With each side of the primal mesh, denoted by $A_j$ (whose length is $|A_j|$), for $j \in [1, J]$, we associate a quadrilateral named "diamond cell" and denoted by $D_j$. When $A_j$ is not on the boundary, this cell is obtained by joining the points $S_{k_1(j)}$ and $S_{k_2(j)}$, which are the two nodes of $A_j$, with the points $G_{i_1(j)}$ and $G_{i_2(j)}$ associated to the elements of the primal mesh which share this side. When $A_j$ is on the boundary $\Gamma$, the cell $D_j$ is obtained by joining the two nodes of $A_j$ with the point $G_{i_1(j)}$ associated to the only element of the primal mesh of which $A_j$ is a side and to the point $G_{i_2(j)}$ associated to $A_j$ (*i.e.* by convention $i_2(j)$ is element of $[I+1, I+J^\Gamma]$ when $A_j$ is located on $\Gamma$). The cells $D_j$ constitute a third partition of $\Omega$, which we name "diamond mesh". The area of the cell $D_j$ is denoted by $|D_j|$. Such a cell is displayed in figure 3.

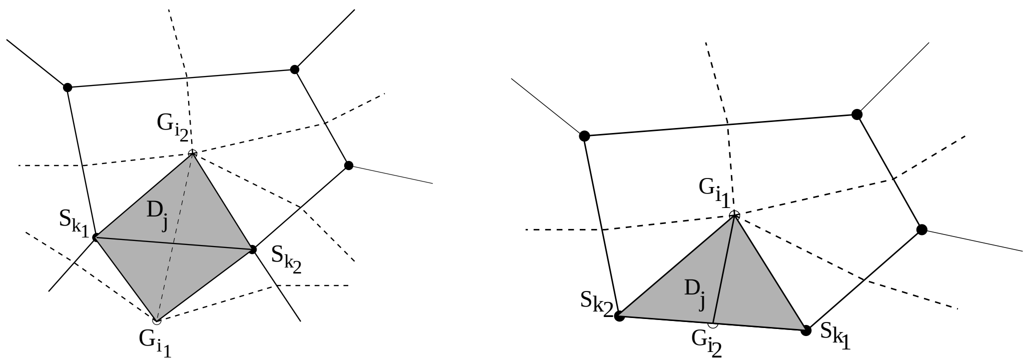


Figure 3: Examples of diamond cells.

### 2.4 Definitions of geometrical elements

The unit vector normal to $A_j$ is denoted by $\mathbf{n}_j$ and is oriented so that $\mathbf{G_{i_1(j)}G_{i_2(j)}} \cdot \mathbf{n}_j \geq 0$. We further denote by $A'_j$ the segment $[G_{i_1(j)}G_{i_2(j)}]$ (whose length is $|A'_j|$) and by $\mathbf{n}'_j$ the unit vector normal to $A'_j$ oriented so that $\mathbf{S_{k_1(j)}S_{k_2(j)}} \cdot \mathbf{n}'_j \geq 0$ (see Fig. 4).
We note by $M_j$ the middle point of $A_j$, $A'_{j1}$ (respectively $A'_{j2}$) the edge $[G_{i_1(j)}M_j]$ (resp. $[M_j G_{i_2(j)}]$) and $\mathbf{n}'_{j1}$ (resp. $\mathbf{n}'_{j2}$) the unit vector normal to $A'_{j1}$ (resp. $A'_{j2}$) oriented such that:

$$|A'_j|\mathbf{n}'_j = |A'_{j1}|\mathbf{n}'_{j1} + |A'_{j2}|\mathbf{n}'_{j2} \,. \tag{6}$$

We define for each $i \in [1, I]$ the set $\mathcal{V}(i)$ of integers $j \in [1, J]$ such that $A_j$ is a side

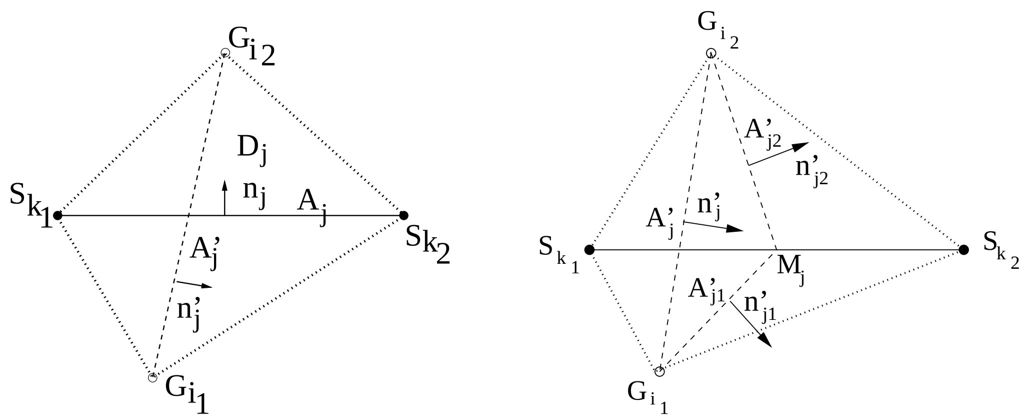


Figure 4: Notations for the diamond cell.

of $T_i$ and for each $k \in [1, K]$ the set $\mathcal{E}(k)$ of integers $j \in [1, J]$ such that $S_k$ is a node of $A_j$.
We define for each $j \in [1, J]$ and each $k$ such that $j \in \mathcal{E}(k)$ (resp. each $i$ such that $j \in \mathcal{V}(i)$) the real-valued number $s'_{jk}$ (resp. $s_{ji}$) whose value is $+1$ or $-1$ whether $\mathbf{n}'_j$ (resp. $\mathbf{n}_j$) points outwards or inwards $P_k$ (resp. $T_i$). We define $\mathbf{n}'_{jk} := s'_{jk}\mathbf{n}'_j$ (resp. $\mathbf{n}_{ji} := s_{ji}\mathbf{n}_j$) and remark that $\mathbf{n}'_{jk}$ (resp. $\mathbf{n}_{ji}$) always points outwards $P_k$ (resp. $T_i$).
At last, for $j \in [1, J - J^\Gamma]$ as on Fig. 5, we denote by $D_{j,1}$ and $D_{j,2}$, the triangles $S_{k_1(j)}G_{i_1(j)}S_{k_2(j)}$ and $S_{k_2(j)}G_{i_2(j)}S_{k_1(j)}$. In the same way, we denote by $D'_{j,1}$ and $D'_{j,2}$, the triangles $G_{i_2(j)}S_{k_1(j)}G_{i_1(j)}$ and $G_{i_1(j)}S_{k_2(j)}G_{i_2(j)}$. In what follows, we shall suppose that all diamond cells are convex so that we can split each interior diamond cell $D_j$ into two triangles in two ways: either $D_j = D_{j,1} \bigcup D_{j,2}$ or $D_j = D'_{j,1} \bigcup D'_{j,2}$.
For diamond cells located on the boundary, let us note that we have $D_{j,1} = D_j$ and $D_{j,2} = \emptyset$. Thus, we have $D_j = D_{j,1} \bigcup D_{j,2} = D'_{j,1} \bigcup D'_{j,2}$.

### 2.5 The discrete operators

We only need the definition of the discrete gradient (resp. divergence) operator on the diamond cells (resp. on the primal and dual cells) to discretize the continuous Laplacian operator. The construction of these two operators is given in [DO 05] but we can find in [DDO 07] the expansion of discrete differential scalar and vector curl

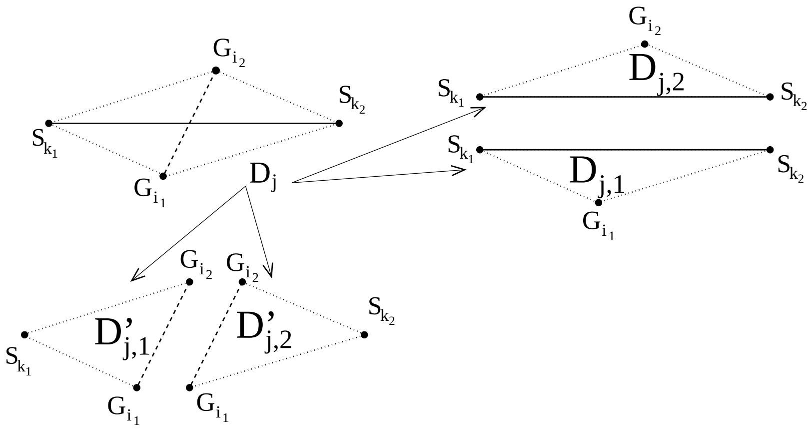


Figure 5: A diamond cell can be split into two distinct ways.

operators defined on the primal, dual and diamond meshes which allow to obtain a Laplacian operator as well.

DEFINITION 2.1 The discrete gradient $\boldsymbol{\nabla}_h^D$ is defined by its values on the diamond cells $D_j$:

$$(\boldsymbol{\nabla}_h^D \phi)_j := \frac{1}{2\,|D_j|}\left\{ \left[\phi_{k_2}^P - \phi_{k_1}^P\right] |A'_j| \mathbf{n}'_j + \left[\phi_{i_2}^T - \phi_{i_1}^T\right] |A_j| \mathbf{n}_j \right\} . \tag{7}$$

Note that formula (7) is exact for an affine function $\phi$ if we set $\phi_{k_\alpha}^P := \phi(S_{k_\alpha})$ and $\phi_{i_\alpha}^T := \phi(G_{i_\alpha})$, for $\alpha \in \{1;2\}$. Computing the discrete gradient only requires the values of $\phi$ at the nodes of the primal and dual meshes. The operator $\boldsymbol{\nabla}_h^D$ thus acts from $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ into $\left(\mathbb{R}^J\right)^2$.
Next, we define the discrete divergence of a vector field $\mathbf{u}$ by its values both on the primal and dual cells of the mesh. Supposing that the vector field $\mathbf{u}$ is given by its discrete values $\mathbf{u}_j$ on the cells $D_j$, we state *the definition* of the discrete divergence $\nabla_h^T \cdot$ on each $T_i$ and the discrete divergence $\nabla_h^P \cdot$ on each $P_k$.

DEFINITION 2.2 The discrete divergence $\nabla_h^{T,P} \cdot := (\nabla_h^T \cdot, \nabla_h^P \cdot)$ is defined by its values over the primal cells $T_i$ and the dual cells $P_k$:

$$\begin{aligned}
(\nabla_h^T \cdot \mathbf{u})_i \; &:= \; \frac{1}{|T_i|} \sum_{j \in \mathcal{V}(i)} |A_j| \mathbf{u}_j \cdot \mathbf{n}_{ji} \; , \\
(\nabla_h^P \cdot \mathbf{u})_k \; &:= \; \frac{1}{|P_k|} \left( \sum_{j \in \mathcal{E}(k)} \left( |A'_{j_1}| \mathbf{n}'_{jk_1} + |A'_{j_2}| \mathbf{n}'_{jk_2} \right) \cdot \mathbf{u}_j \right. \\
&\qquad \left. + \sum_{j \in \mathcal{E}(k) \cap [J - J^\Gamma + 1, J]} \frac{1}{2} \, |A_j| \mathbf{u}_j \cdot \mathbf{n}_j \right) .
\end{aligned} \tag{8}$$

where we recall that $\mathcal{V}(i)$ (resp. $\mathcal{E}(k)$) is the set of integers $j \in [1, J]$ such that $A_j$ (resp. $A'_j$) is a side of $T_i$ (resp. $P_k$) and that $\mathbf{n}_{ji}$ (resp. $\mathbf{n}'_{jk}$) is the unit vector orthogonal to $A_j$ (resp. $A'_j$) pointing outward $T_i$ (resp. $P_k$).

The operator $\nabla_h \cdot$ acts from $\left(\mathbb{R}^J\right)^2$ into $\mathbb{R}^I \times \mathbb{R}^K$. Remark that if the node $S_k$ is not on the boundary $\Gamma$ (*i.e.* if $k \in [1, K - J^\Gamma]$), then the set $\mathcal{E}(k) \cap [J - J^\Gamma + 1, J]$ is empty. On the contrary, if $P_k$ is a boundary dual cell, then the set $\mathcal{E}(k) \cap [J - J^\Gamma + 1, J]$ is composed of the two boundary edges which have $S_k$ as a vertex.
For a given function $\mathbf{u}$, it is straightforward to check that these formulae are the exact mean-values of $\nabla \cdot \mathbf{u}$ over $T_i$ (respectively over an inner $P_k$) if $\mathbf{u}_j \cdot \mathbf{n}_{ji}$ (resp. $\mathbf{u}_j \cdot \mathbf{n}'_{jk}$) represents the mean-value of $\mathbf{u} \cdot \mathbf{n}_{ji}$ over $A_j$ (resp. of $\mathbf{u} \cdot \mathbf{n}'_{jk}$ over $A'_j$). Moreover, for a given vector field $\mathbf{u}$, it is straightforward to check that the formulae (8) are the exact mean-values of $\nabla \cdot \mathbf{u}$ over $T_i$, respectively over inner $P_k$, if

$$|A_j| \mathbf{u}_j \cdot \mathbf{n}_{ji} = \int_{A_j} \mathbf{u} \cdot \mathbf{n}_{ji} \, ds \ , \tag{9}$$

respectively if

$$\left(|A'_{j_1}| \mathbf{n}'_{jk_1} + |A'_{j_2}| \mathbf{n}'_{jk_2}\right) \cdot \mathbf{u}_j = \int_{A'_{j_1}} \mathbf{u} \cdot \mathbf{n}'_{jk_1} \, ds + \int_{A'_{j_2}} \mathbf{u} \cdot \mathbf{n}'_{jk_2} \, ds. \tag{10}$$

Note also that we can replace $\left(|A'_{j1}| \mathbf{n}'_{jk1} + |A'_{j2}| \mathbf{n}'_{jk2}\right)$ by $|A'_j| \mathbf{n}'_{jk}$ since these quantities are equal.

## 2.6 Definitions of the discrete scalar product and norm on the diamond mesh

As will be seen in what follows, we shall associate with each edge $A_j$ ($j \in [1, J]$) discrete values. This leads us to the definition of the following discrete scalar product on the diamond cells.

Definition 2.3 (The discrete scalar product) We define a discrete scalar product on the diamond mesh: for all $(\mathbf{u}, \mathbf{v}) = ((\mathbf{u}_j), (\mathbf{v}_j)) \in \left(\mathbb{R}^{2J}\right) \times \left(\mathbb{R}^{2J}\right)$

$$(\mathbf{u}, \mathbf{v})_D := \sum_{j \in [1,J]} |D_j| \, \mathbf{u}_j \cdot \mathbf{v}_j. \tag{11}$$

Further, for any $\phi \in \mathbb{R}^{I + J^\Gamma} \times \mathbb{R}^K$, we shall define a discrete $H^1$-seminorm on the diamond mesh with the help of the discrete gradient operator defined above (see Eq. (7)):

$$|\phi|_{1,D} := \left(\boldsymbol{\nabla}_h^D \phi, \boldsymbol{\nabla}_h^D \phi\right)_D^{1/2} \ .$$

## 2.7 Discrete Laplacian

In this section, we describe the discrete schemes obtained for the Laplacian problem with the DDFV method. The construction of these schemes is explained in [DO

05]. The Laplacian problem with homogeneous Dirichlet boundary conditions (1) is discretized in the following way

$$\begin{aligned}
-(\nabla_h^T \cdot \boldsymbol{\nabla}_h^D \overline{\phi})_i &= f_i^T,\ \forall i \in [1, I], & (12)\\
-(\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \overline{\phi})_k &= f_k^T,\ \forall k \in [1, K - J^\Gamma], & (13)\\
\overline{\phi}_i^T = \overline{\phi}_k^P &= 0,\ \forall i \in [I+1, I+J^\Gamma], \forall k \in [K - J^\Gamma + 1, K], & (14)
\end{aligned}$$

where $f_i^T$ and $f_k^P$ are the mean-values of $f$ over $T_i$ and $P_k$ defined by:

$$f_i^T = \frac{1}{|T_i|} \int_{T_i} f(\mathbf{x})\ d\mathbf{x} \quad \text{and} \quad f_k^P = \frac{1}{|P_k|} \int_{P_k} f(\mathbf{x})\ d\mathbf{x}.$$

The linear system (12)–(14) has a unique solution $\overline{\phi} \in \overline{V}_D$, with $\overline{V}_D$ defined by:

$$\begin{aligned}
\overline{V}_D := \Big\{ \overline{\phi} = \Big( (\overline{\phi}_i^T), (\overline{\phi}_k^P) \Big) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K : \\
\overline{\phi}_i^T = 0,\ \forall i \in [I+1, I+J^\Gamma] \quad \text{and} \quad \overline{\phi}_k^P = 0,\ \forall k \in [K - J^\Gamma + 1, K] \Big\}.
\end{aligned} \tag{15}$$

The existence and the uniqueness of the solution are proved in [DO 05]. Moreover, [DO 05] shows that this scheme is equivalent to a finite element method and gives the error estimates for continuous solutions in $H^2(\Omega)$. In the same way, the problem (2) with homogeneous Neumann boundary conditions is discretized by

$$\begin{aligned}
-(\nabla_h^T \cdot \boldsymbol{\nabla}_h^D \overline{\phi})_i &= f_i^T,\ \forall i \in [1, I], & (16)\\
-(\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \overline{\phi})_k &= f_k^T,\ \forall k \in [1, K], & (17)\\
(\boldsymbol{\nabla}_h^D \overline{\phi})_j \cdot \mathbf{n}_j &= 0,\ \forall j \in [J - J^\Gamma + 1, J] & (18)\\
\sum_{i \in [1,I]} |T_i|\ \overline{\phi}_i^T = \sum_{k \in [1,K]} |P_k|\ \overline{\phi}_k^P &= 0. & (19)
\end{aligned}$$

The linear system (16)–(19) has a unique solution $\overline{\phi} \in \overline{V}_N$, with $\overline{V}_N$ defined by:

$$\overline{V}_N := \left\{ \overline{\phi} = \Big( (\overline{\phi}_i^T), (\overline{\phi}_k^P) \Big) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K : \sum_{i \in [1,I]} |T_i|\ \overline{\phi}_i^T = \sum_{k \in [1,K]} |P_k|\ \overline{\phi}_k^P = 0 \right\}, \tag{20}$$

which implies $\int_\Omega f(\mathbf{x})\ d\mathbf{x} = 0$. Moreover, these two schemes with non homogeneous boundary conditions are studied in [DDO 07]. In what follows, we need the projection of continuous functions on the discrete space to express the error estimates.

Definition 2.4 We define, for any continuous function $\phi$, the following element $\Pi\phi$ by

$$\begin{aligned}
\forall i \in [1, I + J^\Gamma],\quad (\Pi\phi)_i^T &= \phi(G_i), & (21)\\
\forall k \in [1, K],\quad (\Pi\phi)_k^P &= \phi(S_k). & (22)
\end{aligned}$$

At last, we define below an operator, noted by $\boldsymbol{\delta}$.

Definition 2.5 Let $\phi$ be a continuous function. On each diamond cell $D_j$, we define the constant vector $(\boldsymbol{\delta}\phi)_j$ by the following scalar products:

$$(\boldsymbol{\delta}\phi)_j \cdot \mathbf{n}_j = \frac{1}{|A_j|} \int_{A_j} \boldsymbol{\nabla}\phi \cdot \mathbf{n}_j(\xi)\ d\xi \tag{23}$$

and

$$(\boldsymbol{\delta}\phi)_j \cdot \mathbf{n}'_j = \frac{1}{|A'_j|} \int_{A'_j} \boldsymbol{\nabla}\phi \cdot \mathbf{n}'_j(\xi)\ d\xi. \tag{24}$$

# 3 Error estimate

In section 3.1, we first recall a theorem (theorem 3.1) proved in [DO 05] for smooth solutions on convex domains and then, we state a new theorem (theorem 3.2) which is the analogue of theorem 3.1 but on nonconvex domains for nonsmooth solutions. Subsections 3.2 to 3.4 provide the tools used to prove theorem 3.2 in section 3.5.

## 3.1 Main results

When obtaining error estimates, we will use the following assumption concerning the angle between diagonals of diamond cells (see Fig. 6).

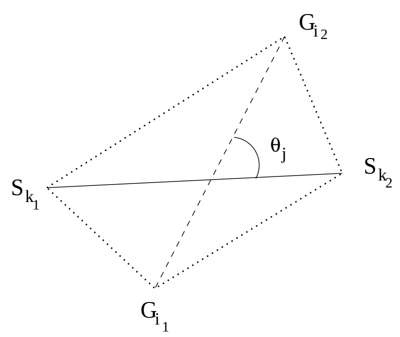


Figure 6: Angle between diagonals of a diamond cell.

Hypothesis 1 The angles between the diagonals of the diamond cells are greater than an angle $\theta^*$ which is strictly positive and independent of the mesh:

$$\exists\ \theta^*,\ 0 < \theta^* < \frac{\pi}{2}\ \ \textit{such that}\ \ \theta_j \geq \theta^*, \forall j \in [1, J]. \tag{25}$$

We estimate the $H^1$-seminorm of the error between $\overline{\phi}$ the element of $\overline{V}_D$ (resp. $\overline{V}_N$) solution of the system (12)–(14) (resp. (16)–(19)) and the projection of the exact solution $\Pi\phi$ (see Definition 2.4). When the domain is convex, denoted by $\Omega_{conv}$, [DO 05] has proved the following theorem for the Laplacian with homogeneous Dirichlet boundary conditions:

THEOREM 3.1 If all diamond cells are convex, $f \in L^2(\Omega_{conv})$, and under Hypothesis 1, there exists a constant $C(\theta^*)$ independent of the step of the mesh $h_{conv}$ such that

$$|\overline{\phi} - \Pi\phi|_{1,D} \le C(\theta^*)\ h_{conv} \|f\|_{L^2(\Omega)}.$$

A similar analysis to [DO 05] allows to obtain an analogous estimate for the discrete Laplacian problem with homogeneous Neumann boundary conditions. However, when the domain $\Omega$ is nonconvex, we have numerically established in [DDO 07] (see also Fig. 8 of section 4) that the order of convergence is non-optimal.
The aim of what follows is to state a similar theorem, for nonsmooth solutions, over nonconvex domains, by using appropriate refinements which allow to restore the optimal order of convergence, and then to prove this theorem.

We shall work on half-diamonds $D_{j,\gamma}$ (resp. $D'_{j,\gamma}$) of Fig. 5 which are supposed to be open and which represent a triangulation $\mathcal{T}_h$ (resp. $\mathcal{T}'_h$) of $\Omega$.
As in Hypothesis 1, we assume that the $D_{j,\gamma}$ (resp. $D'_{j,\gamma}$) cannot degenerate when $h$ tends to 0.

HYPOTHESIS 2 The families $(\mathcal{T}_h)_{h>0}$ and $(\mathcal{T}'_h)_{h>0}$ of triangulations of $\Omega$ are regular in the sense of Ciarlet [CIA 78], which means that

$$\exists\ \sigma > 0 \ \text{ such that } \ \frac{h_{j,\gamma}}{\rho_{j,\gamma}} \ \le\ \sigma,\ \ \forall D_{j,\gamma} \in \mathcal{T}_h,\ \forall h > 0, \tag{26}$$

$$\exists\ \sigma' > 0 \ \text{ such that } \ \frac{h'_{j,\gamma}}{\rho'_{j,\gamma}} \ \le\ \sigma',\ \ \forall D'_{j,\gamma} \in \mathcal{T}'_h,\ \forall h > 0, \tag{27}$$

where $h_{j,\gamma}$ (resp. $h'_{j,\gamma}$) is the diameter of $D_{j,\gamma}$ (resp. $D'_{j,\gamma}$), whereas $\rho_{j,\gamma}$ (resp. $\rho'_{j,\gamma}$) is the diameter of the inscribed circle in $D_{j,\gamma}$ (resp. $D'_{j,\gamma}$). Moreover, we note by $h = \max\limits_{j\in[1,J],\gamma\in\{1,2\}} h_{j,\gamma}$.

Remark that according to [BKP 79] or [Lemma 8.4.1.2, GRI 85], the set $H^{2,\alpha}(\Omega)$ is continuously imbedded into $C^0(\overline{\Omega})$ if $\alpha < 1$, which implies that $\phi \in C^0(\overline{\Omega})$ and then $\Pi\phi$ is meaningful. Now, we have the necessary notations to state theorem 3.2, where we recall that $r(\mathbf{x})$ is the distance from $\mathbf{x}$ to the origin of the domain:

THEOREM 3.2 Let $\phi \in H^{2,\alpha}(\Omega)$, with $\alpha \in [0, \frac{1}{2}[$, the solution of problem (1) or (2). Assume that the diamond cells are convex and verified the Hypothesis 1, then if triangulations $\mathcal{T}_h$ and $\mathcal{T}'_h$ satisfy Hypothesis 2 and following hypotheses of refinement:

$$h_{j,\gamma} \ \le\ \zeta\ h^{1/1-\alpha}, \qquad \text{if } (0,0) \in \overline{D}_{j,\gamma}, \tag{28}$$

$$h_{j,\gamma} \ \le\ \zeta\ h\ \left[\inf_{\mathbf{x}\in D_{j,\gamma}} r(\mathbf{x})\right]^{\alpha}, \quad \text{if } (0,0) \notin \overline{D}_{j,\gamma}, \tag{29}$$

$$h'_{j,\gamma} \ \le\ \zeta\ h^{1/1-\alpha}, \qquad \text{if } (0,0) \in \overline{D}'_{j,\gamma}, \tag{30}$$

$$h'_{j,\gamma} \ \le\ \zeta\ h\ \left[\inf_{\mathbf{x}\in D'_{j,\gamma}} r(\mathbf{x})\right]^{\alpha}, \quad \text{if } (0,0) \notin \overline{D}'_{j,\gamma}, \tag{31}$$

with $\zeta > 0$, there exists $C(\theta^*) > 0$ such that:

$$|\overline{\phi} - \Pi\phi|_{1,D} \le C(\theta^*)\ h\ \ |\phi|_{H^{2,\alpha}(\Omega)}\,. \tag{32}$$

*Remark 1* Hypotheses (28) to (31), defined on half-diamond cells, are the analogue of those used in [RAU 78,GRI 85,DNT 02] on the primal cells.

## 3.2 Preliminary bound

Let $\omega_{j,1}$ (resp. $\omega_{j,2}$) be the first-order interpolating polynomial of $\phi$ on the triangle $D_{j,1}$ (resp. $D_{j,2}$) whose value at each of the three nodes of $D_{j,1}$ (resp. $D_{j,2}$) is equal to the value of the function $\phi$ at this point. The following lemma can be deduced from [DO 05, Lemma 5.9] and also using some ideas of the proof of [DO 05, Lemma 5.10]. These ideas can be applied here because they are independent of the fact that $\Omega$ is convex or not. Let us define on each diamond cell $D_j$ the quantity

$$\mathbf{e}_j(\mathbf{x}) := (\boldsymbol{\delta}\phi)_j - \boldsymbol{\nabla}\phi(\mathbf{x}). \tag{33}$$

Then, we have a first bound of the left-hand side of (32):

LEMMA 3.3 Under Hypothesis 1, for convex diamond cells, we have the following inequality

$$\begin{aligned}|\overline{\phi} - \Pi\phi|_{1,D} \leq \; & \frac{\sqrt{2}}{\sin\theta^*} \left( \sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \left[ \int_{D_{j,\gamma}} (\mathbf{e}_j \cdot \mathbf{n}_j)^2 d\mathbf{x} + \int_{D'_{j,\gamma}} (\mathbf{e}_j \cdot \mathbf{n}'_j)^2 d\mathbf{x} \right] \right)^{1/2} \\ & + \left( \sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \int_{D_{j,\gamma}} |\boldsymbol{\nabla}\phi(\mathbf{x}) - \boldsymbol{\nabla}\omega_{j,\gamma}|^2 \, d\mathbf{x} \right)^{1/2} .\end{aligned} \tag{34}$$

**Proof** We start from [DO 05, Lemma 5.9], which states

$$|\overline{\phi} - \Pi\phi|_{1,D} \leq \left( \sum_{j\in[1,J]} \int_{D_j} |\mathbf{e}_j|^2 d\mathbf{x} \right)^{1/2} + \left( \sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \int_{D_{j,\gamma}} |\boldsymbol{\nabla}\phi(\mathbf{x}) - \boldsymbol{\nabla}\omega_{j,\gamma}|^2 \, d\mathbf{x} \right)^{1/2} .$$

Using the respective scalar products of $\mathbf{e}_j$ with $\mathbf{n}_j$ and $\mathbf{n}'_j$, the term $|\mathbf{e}_j|^2$ can be bounded (see the proof of [Lemma 5.10, DO 05]) by

$$|\mathbf{e}_j|^2 \;\; \leq \;\; \frac{2}{1 - (\mathbf{n}_j \cdot \mathbf{n}'_j)^2} \left[ (\mathbf{e}_j \cdot \mathbf{n}_j)^2 + (\mathbf{e}_j \cdot \mathbf{n}'_j)^2 \right].$$

Moreover, the equality $1 - (\mathbf{n}_j \cdot \mathbf{n}'_j)^2 = (\sin\theta_j)^2 \geq (\sin\theta^*)^2$, under Hypothesis 1, completes this proof. ∎

## 3.3 Results on the reference triangle $\widehat{\mathcal{T}}$

We consider $\widehat{\mathcal{T}}$ the reference triangle whose vertices are $\widehat{S}_1(0,0)$, $\widehat{S}_2(1,0)$ and $\widehat{S}_3(0,1)$. Moreover, we note by $\widehat{A}$ an edge of $\widehat{\mathcal{T}}$.
As $H^{1,\alpha}(\widehat{\mathcal{T}})$ is continuously imbedded into $L^2(\partial\widehat{\mathcal{T}})$ (see [proof of Lemma 3.4, DNT 02]), which is a subset of $L^1(\partial\widehat{\mathcal{T}})$, we can state the following lemma whose proof is quite classical:

LEMMA 3.4 There exists $\widehat{C} > 0$ such that for any $\widehat{v} \in H^{1,\alpha}(\widehat{\mathcal{T}})$, with $\alpha \in [0, 1[$, satisfying $\displaystyle\int_{\widehat{A}} \widehat{v}(\xi)\, d\xi = 0$, we have

$$\|\widehat{v}\|_{L^2(\widehat{\mathcal{T}})} \leq \widehat{C}\ |\widehat{v}|_{H^{1,\alpha}(\widehat{\mathcal{T}})}\,. \tag{35}$$

According to [BKP 79] or [GRI 85, Lemma 8.4.1.2], the set $H^{2,\alpha}(\widehat{\mathcal{T}})$ is continuously imbedded into $C^0(\overline{\widehat{\mathcal{T}}})$ if $\alpha < 1$, which implies that $\widehat{\phi} \in C^0(\overline{\widehat{\mathcal{T}}})$ and $\widehat{\phi}(\widehat{S}_l)$, with $l = 1, 2, 3$, is meaningful. Thus, we can interpolate $\widehat{\phi}$ with a one order polynomial: let $P_1(\widehat{\mathcal{T}})$ be the set of first-order polynomials restricted to $\widehat{\mathcal{T}}$, then for all $\widehat{\phi} \in H^{2,\alpha}(\widehat{\mathcal{T}})$, with $\alpha \in [0, 1[$, there exists a unique $\widehat{\omega} \in P_1(\widehat{\mathcal{T}})$ such that

$$\widehat{\omega}(\widehat{S}_l) = \widehat{\phi}(\widehat{S}_l),\ \ l = 1, 2, 3. \tag{36}$$

In what follows, we need the following lemma, provided by [GRI 85].

LEMMA 3.5 If $\alpha \in [0, 1[$ and $\widehat{\omega}$ defined in (36), then there exists $\widehat{C} > 0$ such that

$$\|\widehat{\phi} - \widehat{\omega}\|_{H^1(\widehat{\mathcal{T}})} \leq \widehat{C}\ |\widehat{\phi}|_{H^{2,\alpha}(\widehat{\mathcal{T}})}, \forall \widehat{\phi} \in H^{2,\alpha}(\widehat{\mathcal{T}}). \tag{37}$$

### 3.4 Similar results over triangle $\mathcal{T}_{j,\gamma}$

After a change of variables, we shall apply the results of section 3.3 to terms of the right-hand side of the inequality (34). We define below the one-to-one mapping from $\widehat{\mathcal{T}}$ into $D_{j,\gamma}$.

DEFINITION 3.6 Let $\mathcal{T}_h$ be the triangulation of $\Omega$, composed of half-diamonds $D_{j,\gamma}$, defined in section 3.1. We consider $D_{j,\gamma} \in \mathcal{T}_h$ whose vertices are $S_l^{j,\gamma}$ with $l = 1, 2, 3$ and the reference triangle $\widehat{\mathcal{T}}$. Then, there exists a one-to-one mapping

$$\Phi_{j,\gamma} : \begin{array}{ccl} \widehat{\mathcal{T}} & \longrightarrow & D_{j,\gamma} \\ (\widehat{x}_1, \widehat{x}_2)^t & \longmapsto & (x_1, x_2)^t = B_{j,\gamma}\,(\widehat{x}_1, \widehat{x}_2)^t + b_{j,\gamma} \end{array}, \tag{38}$$

built such that $\Phi_{j,\gamma}(\widehat{S}_l) = S_l^{j,\gamma},\ l = 1, 2, 3,$ where
$B_{j,\gamma} = (S_2^{j,\gamma} - S_1^{j,\gamma}, S_3^{j,\gamma} - S_1^{j,\gamma})$ is a matrix in $\mathbb{R}^{2\times 2}$ and $b_{j,\gamma} = S_1^{j,\gamma}$ is a vector in $\mathbb{R}^2$.

In the same way, we define a one-to-one mapping $\Phi'_{j,\gamma}$ from $\widehat{\mathcal{T}}$ into $D'_{j,\gamma} \in \mathcal{T}'_h$, using a matrix denoted by $B'_{j,\gamma}$.

PROPOSITION 3.7 Let $\alpha \in [0, \frac{1}{2}[$. Then, there exists $C > 0$ such that for all $\phi \in H^{2,\alpha}(\Omega)$, we have

$$\begin{array}{rcll}
\|\mathbf{e}_j \cdot \mathbf{n}_j\|_{L^2(D_{j,\gamma})} & \leq & C\ \|B_{j,\gamma}^{-1}\|_2^{\alpha}\ \ \|B_{j,\gamma}\|_2\ \ |\phi|_{H^{2,\alpha}(D_{j,\gamma})}\,, & \text{if } (0,0) \in \overline{D}_{j,\gamma} \quad (39) \\
\|\mathbf{e}_j \cdot \mathbf{n}_j\|_{L^2(D_{j,\gamma})} & \leq & C\ \|B_{j,\gamma}\|_2\ \ |\phi|_{H^2(D_{j,\gamma})}\,, & \text{if } (0,0) \notin \overline{D}_{j,\gamma} \quad (40) \\
\|\mathbf{e}_j \cdot \mathbf{n}'_j\|_{L^2(D'_{j,\gamma})} & \leq & C\ \|B'^{-1}_{j,\gamma}\|_2^{\alpha}\ \ \|B'_{j,\gamma}\|_2\ \ |\phi|_{H^{2,\alpha}(D'_{j,\gamma})}\,, & \text{if } (0,0) \in \overline{D'}_{j,\gamma} \quad (41) \\
\|\mathbf{e}_j \cdot \mathbf{n}'_j\|_{L^2(D'_{j,\gamma})} & \leq & C\ \|B'_{j,\gamma}\|_2\ \ |\phi|_{H^2(D'_{j,\gamma})}\,, & \text{if } (0,0) \notin \overline{D'}_{j,\gamma} \quad (42) \\
\|\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\omega_{j,\gamma}\|_{L^2(D_{j,\gamma})} & \leq & C\ \|B_{j,\gamma}^{-1}\|_2^{1+\alpha}\ \ \|B_{j,\gamma}\|_2^2\ \ |\phi|_{H^{2,\alpha}(D_{j,\gamma})}\,, & \text{if } (0,0) \in \overline{D}_{j,\gamma} \quad (43) \\
\|\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\omega_{j,\gamma}\|_{L^2(D_{j,\gamma})} & \leq & C\ \|B_{j,\gamma}^{-1}\|_2\ \ \|B_{j,\gamma}\|_2^2\ \ |\phi|_{H^2(D_{j,\gamma})}\,, & \text{if } (0,0) \notin \overline{D}_{j,\gamma} \quad (44)
\end{array}$$

where $\|.\|_2$ is the Euclidean norm associated to a matrix and $\omega_{j,\gamma} \in P_1(D_{j,\gamma})$ is such that $\omega_{j,\gamma}(S_l^{j,\gamma}) = \phi(S_l^{j,\gamma}), l = 1,2,3$.

**Proof** Using the one-to-one mapping $\Phi_{j,\gamma}$ defined in (38), we verify that

$$\boldsymbol{\nabla}\phi(\mathbf{x}) = (B_{j,\gamma}^t)^{-1}\ \boldsymbol{\nabla}\widehat{\phi}(\widehat{\mathbf{x}}) \tag{45}$$

with $\widehat{\mathbf{x}} = \Phi_{j,\gamma}^{-1}(\mathbf{x})$. While $\phi \in H^{2,\alpha}(D_{j,\gamma})$, then $(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j$ belongs to $H^{1,\alpha}(D_{j,\gamma})$. On the other hand, integral of $(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j$ over $A_j$ is zero. We shall set $v = (\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j$ and do the following change of variables $v(\mathbf{x}) = \widehat{v}(\widehat{\mathbf{x}})$. Thus,

$$\|v\|^2_{L^2(D_{j,\gamma})} = 2\ |D_{j,\gamma}|\ \|\widehat{v}\|^2_{L^2(\widehat{\mathcal{T}})}.$$

After, we can apply Lemma 3.4 to $\widehat{v}$ whose average is zero over $\widehat{A} = \Phi_{j,\gamma}^{-1}(A_j)$ since triangle $D_{j,\gamma}$ is not degenerate. Thus, there exists $C > 0$ such that

$$\|(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j\|^2_{L^2(D_{j,\gamma})} \le C\ |D_{j,\gamma}| \int_{\widehat{\mathcal{T}}} \widehat{r}^{2\alpha}(\widehat{\mathbf{x}})\ |\boldsymbol{\nabla}\widehat{v}|^2 d\widehat{\mathbf{x}}. \tag{46}$$

If $(0,0) \in \overline{D}_{j,\gamma}$, we assume, without loss of generality, that $S_1^{j,\gamma} = (0,0)$, then we can write

$$\widehat{r}(\widehat{\mathbf{x}})\ =\ \|\widehat{\mathbf{x}}\|_2\ =\ \|B_{j,\gamma}^{-1}\mathbf{x}\|_2\ \le\ \|B_{j,\gamma}^{-1}\|_2\ r(\mathbf{x})\ , \tag{47}$$

where $\widehat{r}(\widehat{\mathbf{x}})$ is the distance from $\widehat{\mathbf{x}} \in \widehat{\mathcal{T}}$ to $\widehat{S_1}(0,0)$. By a change of variables in (46), using (47) and (45), we obtain

$$\begin{aligned}\|(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j\|^2_{L^2(D_{j,\gamma})} &\le C\ \|B_{j,\gamma}^{-1}\|_2^{2\alpha} \int_{D_{j,\gamma}} r^{2\alpha}(\mathbf{x})\ |(B_{j,\gamma}^t)\boldsymbol{\nabla}v|^2 d\mathbf{x} \\ &\le C\ \|B_{j,\gamma}^{-1}\|_2^{2\alpha}\ \|B_{j,\gamma}^t\|_2^2 \int_{D_{j,\gamma}} r^{2\alpha}(\mathbf{x})\ |\boldsymbol{\nabla}v|^2 d\mathbf{x},\end{aligned}$$

which can be rewritten as $\|B_{j,\gamma}\|_2 = \|B_{j,\gamma}^t\|_2$,

$$\|(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j\|^2_{L^2(D_{j,\gamma})}\ \le\ C\ \|B_{j,\gamma}^{-1}\|_2^{2\alpha}\ \|B_{j,\gamma}\|_2^2\ |v|^2_{H^{1,\alpha}(D_{j,\gamma})}.$$

Replacing $v$ by its expression and since $(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j$ is a constant over $D_{j,\gamma}$ and $\mathbf{n}_j$ is a unit vector, previous inequality finally gives

$$\|(\boldsymbol{\delta}\phi)\cdot\mathbf{n}_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}_j\|^2_{L^2(D_{j,\gamma})}\ \le\ C\ \|B_{j,\gamma}^{-1}\|_2^{2\alpha}\ \|B_{j,\gamma}\|_2^2\ |\phi|^2_{H^{2,\alpha}(D_{j,\gamma})}, \tag{48}$$

and we deduce (39). Obviously, in the same way with $(\boldsymbol{\delta}\phi)\cdot\mathbf{n}'_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}'_j$ on $D'_{j,\gamma}$, we deduce the existence of $C > 0$ such that

$$\|(\boldsymbol{\delta}\phi)\cdot\mathbf{n}'_j - \boldsymbol{\nabla}\phi\cdot\mathbf{n}'_j\|^2_{L^2(D'_{j,\gamma})} \le C\ \|B'^{-1}_{j,\gamma}\|_2^{2\alpha}\ \|B'_{j,\gamma}\|_2^2\ |\phi|^2_{H^{2,\alpha}(D'_{j,\gamma})}, \tag{49}$$

which implies (41). On the other hand, by a change of variables, then using Lemma 3.5 and $\|B_{j,\gamma}\|_2 = \|B_{j,\gamma}^t\|_2$, we obtain

$$\begin{aligned}\int_{D_{j,\gamma}} |\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\omega_{j,\gamma}|^2\, d\mathbf{x} &= 2\ |D_{j,\gamma}| \int_{\widehat{\mathcal{T}}} \left|(B_{j,\gamma}^t)^{-1}\left(\boldsymbol{\nabla}\widehat{\phi} - \boldsymbol{\nabla}\widehat{\omega}\right)\right|^2 d\widehat{\mathbf{x}} \\ &\le 2\ |D_{j,\gamma}|\ \|(B_{j,\gamma}^{-1})^t\|_2^2\ \|\widehat{\phi} - \widehat{\omega}\|^2_{H^1(\widehat{\mathcal{T}})} \\ &\le C\ |D_{j,\gamma}|\ \|B_{j,\gamma}^{-1}\|_2^2 \left|\widehat{\phi}\right|^2_{H^{2,\alpha}(\widehat{\mathcal{T}})}.\end{aligned} \tag{50}$$

Now, we use the Hessian matrix and the Schur norm which satisfy $\sum_{|\beta|=2} \left| D^{\beta}\widehat{\phi} \right|^2 = \|\widehat{H}(\widehat{\phi})\|_S^2$, then we have

$$\left|\widehat{\phi}\right|^2_{H^{2,\alpha}(\widehat{\mathcal{T}})} \quad = \quad \int_{\widehat{\mathcal{T}}} \widehat{r}^{2\alpha}(\widehat{\mathbf{x}}) \; \|\widehat{H}(\widehat{\phi})\|_S^2 \; d\widehat{\mathbf{x}}.$$

Moreover, $\|B_{j,\gamma}\|_2 = \|B^t_{j,\gamma}\|_2$ and in finite dimension, the norms are equivalent, which implies the existence of two real numbers $C_1 > 0$ and $C_2 > 0$ such that

$$\|\widehat{H}(\widehat{\phi})\|_S \; = \; \|B^t_{j,\gamma} H(\phi) B_{j,\gamma}\|_S \; \leq \; C_1 \; \|B_{j,\gamma}\|_2^2 \; \; \|H(\phi)\|_2 \; \leq \; C_2 \; \|B_{j,\gamma}\|_2^2 \; \; \|H(\phi)\|_S,$$

and by changing variables, we obtain from the previous line and from (47) that

$$\int_{\widehat{\mathcal{T}}} \widehat{r}^{2\alpha} \; \|\widehat{H}(\widehat{\phi})\|_S^2 \; d\widehat{\mathbf{x}} \leq \frac{C_2^2}{2\;|D_{j,\gamma}|} \; \|B^{-1}_{j,\gamma}\|_2^{2\alpha} \; \|B_{j,\gamma}\|_2^4 \; \int_{D_{j,\gamma}} r^{2\alpha}(\mathbf{x}) \; \|H(\phi)\|_S^2 \; d\mathbf{x},$$

which can be rewritten as

$$\left|\widehat{\phi}\right|^2_{H^{2,\alpha}(\widehat{\mathcal{T}})} \leq \frac{C_2^2}{2\;|D_{j,\gamma}|} \|B^{-1}_{j,\gamma}\|_2^{2\alpha} \; \|B_{j,\gamma}\|_2^4 \; \; |\phi|^2_{H^{2,\alpha}(D_{j,\gamma})}\,, \tag{51}$$

and, with (50), we obtain (43). Now, if $(0,0) \notin \overline{D}_{j,\gamma}$, then $H^{2,\alpha}(D_{j,\gamma}) = H^2(D_{j,\gamma})$ in (51) and we can choose $\alpha = 0$, which implies the following estimate

$$\left|\widehat{\phi}\right|^2_{H^{2}(\widehat{\mathcal{T}})} \leq \frac{C_2^2}{2\;|D_{j,\gamma}|} \; \|B_{j,\gamma}\|_2^4 \; \; |\phi|^2_{H^{2}(D_{j,\gamma})}\,. \tag{52}$$

We obtain (44) using (50) and (52). In the same way, if $(0,0) \notin \overline{D}_{j,\gamma}$ (resp. $(0,0) \notin \overline{D'}_{j,\gamma}$), we can choose $\alpha = 0$ in (48) (resp. (49)), which gives (40) (resp. (42)). ■

### 3.5 Proof of theorem 3.2

Let us set $\alpha_{j,\gamma} = \alpha$ if $(0,0) \in \overline{D}_{j,\alpha}$ and $\alpha_{j,\gamma} = 0$ otherwise. In the same way, let us set $\alpha'_{j,\gamma} = \alpha$ if $(0,0) \in \overline{D}'_{j,\alpha}$ and $\alpha'_{j,\gamma} = 0$ otherwise. Applying the proposition 3.7, there exists $C > 0$ such that

$$\sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \|\mathbf{e}_j \cdot \mathbf{n}_j\|^2_{L^2(D_{j,\gamma})} \leq C \sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \|B^{-1}_{j,\gamma}\|_2^{2\alpha_{j,\gamma}} \; \; \|B_{j,\gamma}\|_2^2 \, |\phi|^2_{H^{2,\alpha_{j,\gamma}}(D_{j,\gamma})}\,, \tag{53}$$

$$\sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \|\mathbf{e}_j \cdot \mathbf{n}'_j\|^2_{L^2(D'_{j,\gamma})} \leq C \sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \|{B'}^{-1}_{j,\gamma}\|_2^{2\alpha'_{j\gamma}} \; \; \|B'_{j,\gamma}\|_2^2 \, |\phi|^2_{H^{2,\alpha'_{j,\gamma}}(D'_{j,\gamma})}\,, \tag{54}$$

for all $\gamma \in \{1,2\}$. On the other hand, there also exists $C > 0$ such that

$$\sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \|\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\omega_{j,\gamma}\|^2_{L^2(D_{j,\gamma})} \leq C \sum_{j\in[1,J]} \sum_{\gamma=1}^{2} \|B^{-1}_{j,\gamma}\|_2^{2+2\alpha_{j,\gamma}} \|B_{j,\gamma}\|_2^4 \, |\phi|^2_{H^{2,\alpha_{j,\gamma}}(D_{j,\gamma})}\,. \tag{55}$$

Applying Hypothesis 2, the Euclidean norm $\|.\|_2$ of the matrices $B_{j,\gamma}$ and $B'_{j,\gamma}$, and their inverses can be bounded by (see [theorem 3.1.3, CIA 78])

$$\|B_{j,\gamma}\|_2 \leq \sqrt{2}\ h_{j,\gamma} \quad , \quad \|B'_{j,\gamma}\|_2 \leq \sqrt{2}\ h'_{j,\gamma}\,,$$

$$\|B^{-1}_{j,\gamma}\|_2 \leq \frac{\sqrt{2}}{\rho_{j,\gamma}} \leq \frac{\sqrt{2}\ \sigma}{h_{j,\gamma}} \qquad , \quad \|B'^{-1}_{j,\gamma}\|_2 \leq \frac{\sqrt{2}}{\rho'_{j,\gamma}} \leq \frac{\sqrt{2}\ \sigma'}{h'_{j,\gamma}}\,,$$

which implies, using Hypothesis 2, that (53), (54) and (55) can be rewritten as

$$\begin{aligned}
\sum_{j\in[1,J]}\sum_{\gamma=1}^{2}\|\mathbf{e}_j\cdot\mathbf{n}_j\|^2_{L^2(D_{j,\gamma})} &\leq C\sum_{j\in[1,J]}\sum_{\gamma=1}^{2} h_{j,\gamma}^{2-2\alpha_{j,\gamma}}\ |\phi|^2_{H^{2,\alpha_{j,\gamma}}(D_{j,\gamma})}\,, &(56)\\
\sum_{j\in[1,J]}\sum_{\gamma=1}^{2}\|\mathbf{e}_j\cdot\mathbf{n}'_j\|^2_{L^2(D'_{j,\gamma})} &\leq C\sum_{j\in[1,J]}\sum_{\gamma=1}^{2} {h'}_{j,\gamma}^{2-2\alpha'_{j,\gamma}}\ |\phi|^2_{H^{2,\alpha'_{j,\gamma}}(D'_{j,\gamma})}\,, &(57)\\
\sum_{j\in[1,J]}\sum_{\gamma=1}^{2}\|\boldsymbol{\nabla}\phi-\boldsymbol{\nabla}\omega_{j,\gamma}\|^2_{L^2(D_{j,\gamma})} &\leq C\sum_{j\in[1,J]}\sum_{\gamma=1}^{2} h_{j,\gamma}^{2-2\alpha_{j,\gamma}}\ |\phi|^2_{H^{2,\alpha_{j,\gamma}}(D_{j,\gamma})}\,. &(58)
\end{aligned}$$

We can split the right-hand sides of (56) and (58) (resp. (57)) into two sums according to $(0,0)\in\overline{D}_{j,\gamma}$ (resp. $(0,0)\in\overline{D}'_{j,\gamma}$) or not.
If $(0,0)\in\overline{D}_{j,\gamma}$, then $\alpha_{j,\gamma}=\alpha$. Therefore, Hypothesis (28) implies:

$$h_{j,\gamma}^{2-2\alpha}\leq\zeta^{2-2\alpha}\ h^2.$$

In the same way, if $(0,0)\in\overline{D}'_{j,\gamma}$, then $\alpha'_{j,\gamma}=\alpha$ and applying Hypothesis (30), we obtain

$${h'}_{j,\gamma}^{2-2\alpha}\leq\zeta^{2-2\alpha}\ h^2.$$

If $(0,0)\notin\overline{D}_{j,\gamma}$, then $\alpha_{j,\gamma}=0$ and $H^{2,0}(\Omega)=H^2(\Omega)$. Therefore, introducing $\alpha\neq 0$, Hypothesis (29) implies:

$$\begin{aligned}
h^2_{j,\gamma}\ |\phi|^2_{H^2(D_{j,\gamma})} &= h^2_{j,\gamma}\int_{D_{j,\gamma}} r^{-2\alpha}\ r^{2\alpha}\ \|H(\phi)\|^2_S\ d\mathbf{x}\\
&\leq h^2_{j,\gamma}\left[\inf_{x\in D_{j,\gamma}} r(\mathbf{x})\right]^{-2\alpha}\int_{D_{j,\alpha}} r^{2\alpha}\ \|H(\phi)\|^2_S\ d\mathbf{x}\\
&\leq \zeta^2\ h^2\ |\phi|^2_{H^{2,\alpha}(D_{j,\gamma})}\,.
\end{aligned}$$

In the same way, we deduce from Hypothesis (31) that

$${h'}^2_{j,\gamma}\ |\phi|^2_{H^2(D'_{j,\gamma})}\leq\ \zeta^2\ h^2\ |\phi|^2_{H^{2,\alpha}(D'_{j,\gamma})}.$$

Consequently, the right-hand sides of (56), (57) and (58) are bounded by

$$\sum_{j\in[1,J]}\sum_{\gamma=1}^{2} h_{j,\gamma}^{2-2\alpha_{j,\gamma}}\ |\phi|^2_{H^{2,\alpha_{j,\gamma}}(D_{j,\gamma})}\ \leq\ \max\{\zeta^{2-2\alpha},\zeta^2\}\ h^2\ |\phi|^2_{H^{2,\alpha}(\Omega)}.$$

$$\sum_{j\in[1,J]}\sum_{\gamma=1}^{2} {h'}_{j,\gamma}^{2-2\alpha'_{j,\gamma}} \; |\phi|^2_{H^{2,\alpha'_{j,\gamma}}(D'_{j,\gamma})} \quad \leq \quad \max\{\zeta^{2-2\alpha},\zeta^2\}\; h^2 \; |\phi|^2_{H^{2,\alpha}(\Omega)}.$$

Finally, we close the proof using Lemma 3.3 and combining the previous line with (56), (57) and (58).

# 4 Numerical results

The domain of computation is $\Omega = ]-1;1[^2\backslash]0;1[^2$, so that $\Omega$ has a nonconvex corner at (0,0) with interior angle $\omega = \frac{3\pi}{2}$. The data and boundary conditions are chosen so that the analytic solutions $\psi$ of the Laplacian with Dirichlet boundary conditions and $\phi$ of the Laplacian with Neumann boundary conditions, expressed in polar coordinates centered on $(0,0)$, are given by

$$\psi(r,\theta) = r^{2/3}\sin\left(\frac{2}{3}\theta\right) \quad \text{and} \quad \phi(r,\theta) = r^{2/3}\cos\left(\frac{2}{3}\theta\right) + c,$$

with c a real number such that $\int_\Omega \phi = 0$. These functions correspond to the singular parts $\phi_c$ defined previously in (4) and (5) but extended to the full domain $\Omega$. We notice that $\psi$ and $\phi$ belong to $H^1(\Omega)$ but are not in $H^2(\Omega)$. More precisely, $\psi$ and $\phi$ belong to $\mathrm{H}^{1+\mathrm{s}}(\Omega)$ with $s < \frac{\pi}{\omega}$, in other words $s < 2/3$ here (see [GRI 92] for more explications). In what follows, we evaluate the discrete error in the $H^1$-seminorm on the diamond cells defined by:

$$e^2(h) := \frac{\displaystyle\sum_{j\in[1,J]} |D_j|\;|(\nabla_h^D\overline{\phi})_j - (\nabla_h^D\Pi\phi)_j|^2}{\displaystyle\sum_{j\in[1,J]} |D_j|\;|(\nabla_h^D\Pi\phi)_j|^2},$$

where $\phi$ and $\overline{\phi}$ respectively are the continuous and the numerical solutions.

## 4.1 Unstructured meshes without local refinement

First, we use a family of five unstructured triangular grids. The first two meshes of this family are displayed in figure 7, while the error curves of $\nabla\psi$ and $\nabla\phi$ in the discrete $L^2$-norm are shown in figure 8, together with a reference line of slope 2/3. The order of convergence of the scheme seems to be 2/3 in this case, as in [BP 04, DDO 07].

## 4.2 Structured meshes with local appropriate refinement

For the second family of meshes, we follow the construction of the refinement described in [RAU 78] and [GRI 85]. At first, we divide $\Omega$ into coarse triangles (in our case, there are six structured triangles). Then, each of the triangles of which (0,0) is not a vertex is divided into $n^2$ triangles with $n = 2, 4, 8, 16, 32$ in a uniform way. At last, the triangles which have a vertex at (0,0) are divided in the following way: the sides having (0,0) as an extremity are divided according to the ratios

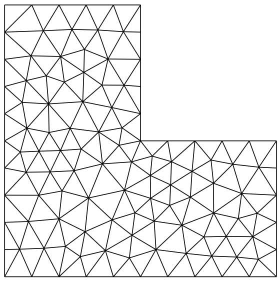

Figure 7: Unstructured meshes.

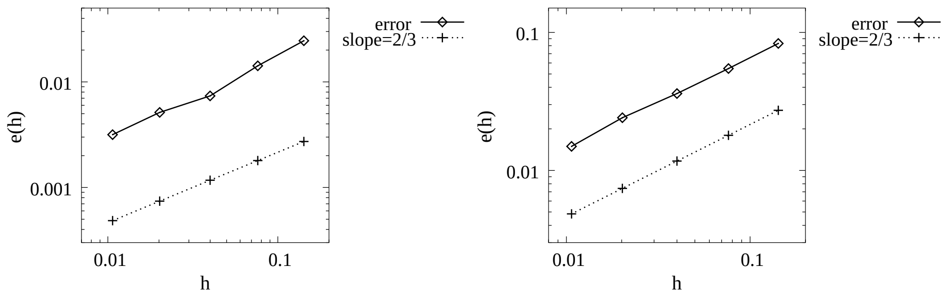


Figure 8: Errors of $\nabla\psi$ and $\nabla\phi$ in the $L^2$ norm for the unstructured meshes.

$\left(\frac{i}{n}\right)^{\frac{1}{(1-\alpha)}}$, $i = 1, ..., n$, while the third side is divided in $n$ subsegments of the same length. For each $i = 1, ..., n$, we join the points $\left(\frac{i}{n}\right)^{\frac{1}{(1-\alpha)}}$ belonging to the two sides having (0,0) as an extremity with a line divided into $i$ subsegments of the same length. Then, we join the different points as on Fig. 9.

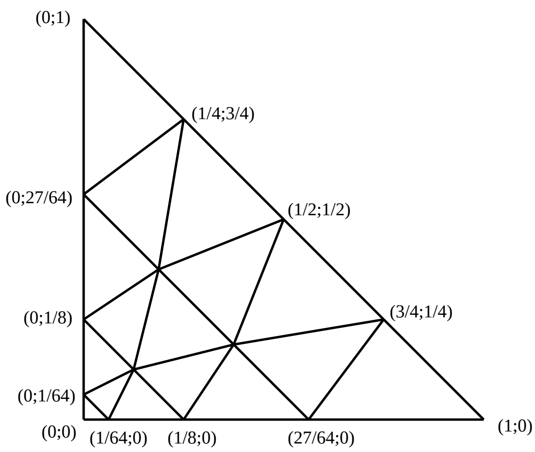


Figure 9: Construction of an $\alpha$-refined triangle whose vertices $(0, 0), (1, 0)$ and $(0, 1)$ with $n = 4$.

Thus, by construction, the primal mesh, called $\alpha$-refined mesh, is regular in the sense of [CIA 78] and satisfies the hypotheses

$$h_i \leq \zeta\, h^{1/1-\alpha}, \qquad \text{if } (0,0) \in \overline{T}_i, \tag{59}$$

$$h_i \leq \zeta\, h \left[\inf_{\mathbf{x}\in T_i} r(\mathbf{x})\right]^{\alpha}, \quad \text{if } (0,0) \notin \overline{T}_i, \tag{60}$$

(see [RAU 78, GRI 85]), where we note by $h_i$ the diameter of $T_i$. Remark that $h$ defined in Hypothesis 2 verifies $h = \max_{i\in[1,I]} h_i$. Moreover, note by $\rho_i$ the diameter of the inscribed circle of $T_i$.

Now, it remains to check that triangles $D_{j,\gamma}$ and $D'_{j,\gamma}$, $\forall j \in [1,J], \forall\gamma \in \{1,2\}$ satisfy hypotheses of Theorem 3.2:
(a) Convexity. By construction, two adjacent cells of the $\alpha$-refined mesh make up a convex quadrilateral (knowing coordinates of points, we can compute equations of diagonals of primal cells and check they intersect inside these cells), so that the diamond cell $D_j$ which is located inside of this quadrilateral is necessarily convex.
(b) Hypotheses of refinement. As $G_i$ is the center of gravity of $T_i$ and a vertex of $D_{j,\gamma}$, we have obviously $h_{j,\gamma} \leq h_i$ and $h'_{j,\gamma} \leq h_{j,1} + h_{j,2}$. Consequently, as the $\alpha$-refined mesh verifies the hypotheses (59) and (60), it implies that the half-diamonds $D_{j,\gamma}$ and $D'_{j,\gamma}$ contained in this mesh satisfy the hypotheses (28) to (31) (if $(0,0) \in T_i$, then for each triangle $D_{j,\gamma} \subset T_i$ such that $(0,0) \notin D_{j,\gamma}$, we have $h_{j,\gamma} \leq c\ \left(\frac{1}{n}\right)^{\frac{1}{1-\alpha}}$, where $c > 0$ is a constant. On the other hand, $\left(\frac{1}{n}\right)^{\frac{1}{1-\alpha}} = \left(\frac{1}{n}\right)\cdot\left(\frac{1}{n}\right)^{\frac{\alpha}{1-\alpha}} \leq C\ h \left[\inf_{\mathbf{x}\in T_i} r(\mathbf{x})\right]^{\alpha}$, with $C > 0$).
(c) Hypotheses 1 and 2. Let $D_{j,\gamma}$ be an half-diamond contained in a triangle $T_i$ of the $\alpha$-refined mesh. As $G_i$ is the center of gravity of $T_i$, then $|T_i| = 3\ |D_{j,\gamma}|$. On the other hand, Heron formula in triangle $D_{j,\gamma}$ gives: $|D_{j,\gamma}| = \rho_{j,\gamma}\frac{\text{perimeter of } D_{j,\gamma}}{2}$, and in $T_i$ we have also: $|T_i| = \rho_i\frac{\text{perimeter of } T_i}{2}$. Since perimeter of $D_{j,\gamma}$ is smaller than perimeter of $T_i$, we deduce that $\rho_i \leq 3\ \rho_{j,\gamma}$. Combining this inequality with $h_{j,\gamma} \leq h_i$, then the regularity in the sense of [CIA 78] of $\alpha$-refined mesh implies (26). In other words, according to [Hypothesis (3.1.43), CIA 78], angles of half-diamonds $D_{j,\gamma}$ are all greater than an angle $\theta_0$. Since primal and diamond cells are convex, we deduce angles of half-diamonds $D'_{j,\gamma}$ are also bounded, which implies (27) and angles of the half-diamonds $D'_{j,\gamma}$ are all greater than an angle $\theta_1$. Finally, we conclude that the angle $\theta_j$ of figure 6 is bounded down by $\theta^*$ independent of mesh. Thus, the diamond cells $D_j$ contained in this mesh satisfy Hypothesis 1.

Figure 10 displays the meshes obtained for $n = 4$ and $n = 8$. The error curves of $\boldsymbol{\nabla}\psi$ and $\boldsymbol{\nabla}\phi$ are shown in figure 11, together with a reference line of slope 1. It seems the order of convergence of the scheme is a little more than 1 in both cases.

### 4.3 Unstructured meshes with local appropriate refinement

At last, we also test the method on a third family which is only locally $\alpha$-refined: the three coarse triangles having (0,0) as a vertex are refined like those built in the second family with $n = 2, 4, 8, 16$, then the rest of the domain $\Omega$ is composed

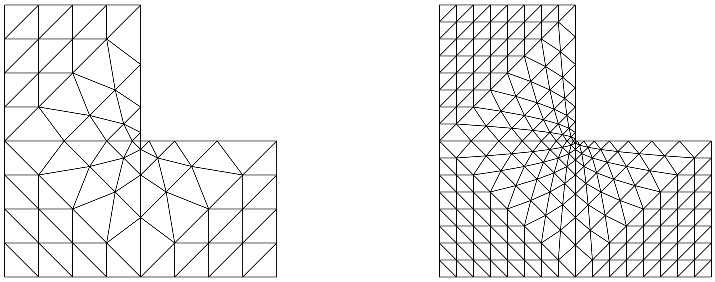

Figure 10: $\alpha$-refined meshes with n=4 and n=8.

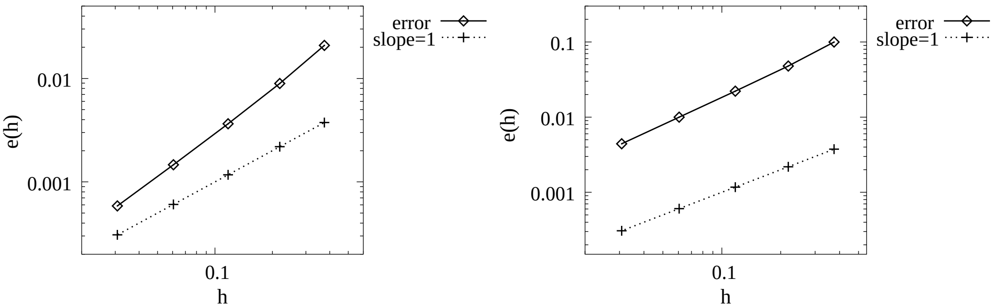


Figure 11: Errors of $\nabla\psi$ and $\nabla\phi$ in the $L^2$-norm for the $\alpha$-refined meshes.

of unstructured triangles. This family of meshes is particularly interesting because it shows a local refinement is enough to obtain the optimal order of convergence, without other constraint on the mesh. Thus, it proves the origin of the loss of convergence order is the singularity. Figure 12 displays the first two meshes of this family. The order of convergence of the scheme seems to be 1 for $\nabla\psi$ and $\nabla\phi$ according to figure 13.

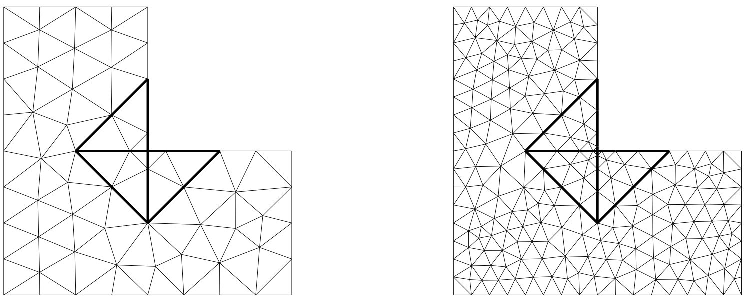

Figure 12: Unstructured $\alpha$-refined meshes with n=2 and n=4.

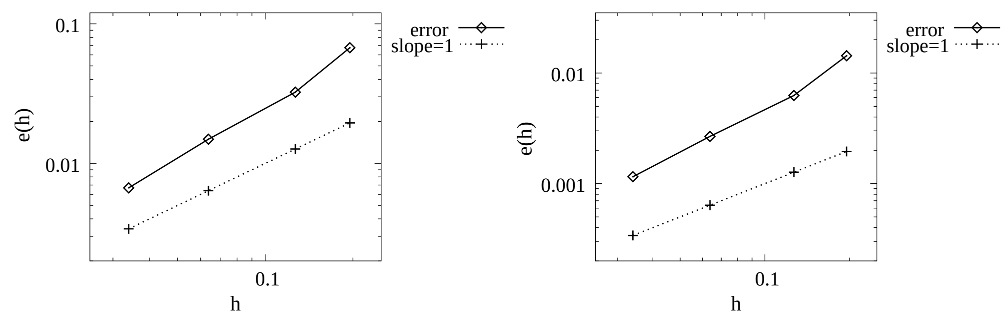


Figure 13: Errors of $\nabla\psi$ and $\nabla\phi$ in the $L^2$-norm for the unstructured $\alpha$-refined meshes

# 5 Conclusion

Many elliptic problems include the Laplacian, such as convection-diffusion problems, fluid problems or div-curl problems. We have shown in this study that when the solution of the Laplacian problem is singular, then we numerically observe a loss of accuracy. However, we have proved that a local refinement allows us to restore the optimal rate of convergence. Moreover, we can apply directly the results of this study to the div-curl problems [DDO 07].

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